

SINGULARITIES OF PLURI-THETA DIVISORS IN CHAR $p > 0$

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ABSTRACT. We show that if (X, Θ) is a PPAV over an algebraically closed field of characteristic $p > 0$ and $D \in |m\Theta|$, then $(X, \frac{1}{m}D)$ is a limit of strongly F -regular pairs and in particular $\text{mult}_x(\bar{D}) \leq m \cdot \dim X$ for any $x \in X$.

1. INTRODUCTION

Let (X, Θ) be a principally polarized abelian variety (PPAV) so that X is a connected projective algebraic group and Θ is an ample divisor with $h^0(X, \mathcal{O}_X(\Theta)) = 1$. The geometry of X is often studied in terms of the singularities of the theta divisor Θ (or more generally of the singularities of pluri-theta divisors i.e. divisors in $|m\Theta|$). For PPAVs over \mathbb{C} there are a number of well known results saying that the singularities of pluri-theta divisors are mild (see for example [Kollar95], [EL97] and [Hacon99]). According to a result of Ein and Lazarsfeld (cf. [EL97, 3.5]), if $D \in |m\Theta|$, then $(X, \frac{1}{m}D)$ is log canonical. Since X is smooth, this is equivalent to saying that $(X, \frac{1-\epsilon}{m}D)$ is Kawamata log terminal for any $0 < \epsilon < 1$. The purpose of this brief note is to prove the analogous result in characteristic $p > 0$.

Theorem 1.1. *Let (X, Θ) be a PPAV over an algebraically closed field of characteristic $p > 0$. If $D \in |m\Theta|$, then $(X, \frac{1-\epsilon}{m}D)$ is strongly F -regular for any rational number $0 < \epsilon < 1$.*

Remark 1.2. In particular it follows that $(X, \frac{1}{m}D)$ is log canonical in the sense that all log-discrepancies are ≥ 0 and therefore $\text{mult}_x(D) \leq m \cdot \dim X$ for any $x \in X$. Note that in characteristic 0 it is known that if $\text{mult}_x(D) = m \cdot \dim X$, then X is a product of elliptic curves [EL97] and [Hacon99]. We do not know if the analogous result holds in characteristic $p > 0$. By [Hernandez11, 4.1] it follows that (X, D) is F -pure, and if p and m are coprime, then (X, D) is sharply F -pure.

Remark 1.3. Since in characteristic 0 it is known that irreducible Θ divisors are normal with rational singularities, it is natural to wonder if over an algebraically closed field of characteristic $p > 0$, irreducible Θ

divisors are normal with F -rational singularities. Note that a related result appears in [BBE07].

The characteristic 0 argument of Ein and Lazarsfeld relies on the theory of multiplier ideal sheaves, Kawamata-Viehweg vanishing and the Fourier-Mukai functor. In characteristic $p > 0$, the theory of Fourier-Mukai functors still applies and multiplier ideal sheaves can be replaced by test ideals. However, there is no good substitute for Kawamata-Viehweg vanishing (which is known to fail in this context). Instead, inspired by some ideas contained in [Schwede11], we use the "generic vanishing results" from [CH03], [Hacon04] and [PP08] to show that $h^0(X, \sigma(X, \frac{1-\epsilon}{m}D) \otimes \mathcal{O}_X(\Theta) \otimes P_{\hat{x}}) > 0$ for a general $\hat{x} \in \hat{X}$. But then a general translate of Θ vanishes along the cosupport of $\sigma(X, \frac{1-\epsilon}{m}D)$. This is only possible if $\sigma(X, \frac{1-\epsilon}{m}D) = \mathcal{O}_X$ and so $(X, \frac{1-\epsilon}{m}D)$ is strongly F -regular and (1.1) follows.

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2. PRELIMINARIES

Throughout this paper we work over an algebraically closed field k of characteristic $p > 0$. Recall that the ring homomorphism $F : k \rightarrow k$ defined by $F(x) = x^p$ endows k with a non-trivial k -module structure.

2.1. Test ideals. Here we recall the definition of test ideals and some related results that will be needed in this paper. We refer the reader to [BST11] and [Schwede11] (and the references therein) for a more complete treatment. Let $(X, \Delta = \sum d_i D_i)$ be a log pair so that X is a normal variety and $\Delta \geq 0$ is a \mathbb{Q} -divisor such that $K_X + \Delta$ is \mathbb{Q} -Cartier. Let $F : X \rightarrow X$ be the Frobenius morphism and for any integer $e > 0$, let F^e be its e -th iterate. The **parameter test submodule** of (X, Δ) denoted by $\tau(\omega_X, \Delta)$ is locally defined as the unique smallest non-zero \mathcal{O}_X -submodule of ω_X such that $\phi(F_*^e M) \subset M$ for any $e > 0$ and any $\phi \in \text{Hom}_{\mathcal{O}_X}(F_*^e \omega_X(\lceil (p^e - 1)\Delta \rceil), \omega_X)$. The **test ideal** $\tau(X, \Delta)$ is defined by $\tau(\omega_X, K_X + \Delta)$. It is known that $\tau(X, \Delta) \subset \mathcal{O}_X$ is an ideal sheaf such that

$$\tau(X, \Delta + A) = \tau(X, \Delta) \otimes \mathcal{O}_X(-A), \quad \text{and} \quad \tau(X, \Delta + eA) = \tau(X, \Delta)$$

for any Cartier divisor A and any rational number $0 \leq e \ll 1$. We also have that test ideals are contained in multiplier ideals in the sense that if $\pi : Y \rightarrow X$ is a proper birational morphism, then $\tau(X, \Delta) \subset$

$\pi_* \mathcal{O}_Y(K_Y - \lfloor \pi^*(K_X + \Delta) \rfloor)$. (Recall that in characteristic 0, if π is a log resolution, then the multiplier ideal is defined by $\mathcal{J}(X, \Delta) = \pi_* \mathcal{O}_Y(K_Y - \lfloor \pi^*(K_X + \Delta) \rfloor)$.) In particular, if X is a smooth variety and $\text{mult}_x(\Delta) \geq \dim X$, then $\tau(X, \Delta) \subset \mathfrak{m}_x$ where \mathfrak{m}_x is the maximal ideal of x in X .

Suppose now that p does not divide the index of $K_X + \Delta$ so that $(p^e - 1)(K_X + \Delta)$ is Cartier for some integer $e > 0$. Let $\mathcal{L}_{e, \Delta} = \mathcal{O}_X((1 - p^e)(K_X + \Delta))$. There is a canonically determined (up to unit) homomorphism of line bundles $\phi_\Delta : F_*^e \mathcal{L}_{e, \Delta} \rightarrow \mathcal{O}_X$. We have that $\tau(X, \Delta)$ is the smallest non-zero ideal $J \subset \mathcal{O}_X$ such that $\phi_\Delta(F_*^e(J \cdot \mathcal{L}_{e, \Delta})) = J$. Similarly, we define $\sigma(X, \Delta)$ to be the largest ideal $J \subset \mathcal{O}_X$ such that $\phi_\Delta(F_*^e(J \cdot \mathcal{L}_{e, \Delta})) = J$. By definition (X, Δ) is **strongly F -regular** if $\tau(X, \Delta) = \mathcal{O}_X$, and **sharply F -pure** if $\sigma(X, \Delta) = \mathcal{O}_X$. By [TW04, 2.2], we have that 1) if (X, Δ) is strongly F -regular then it is also sharply F -pure, and 2) if (X, Δ) is sharply F -pure and X is strongly F -regular, then $(X, (1 - \epsilon)\Delta)$ is strongly F -regular for any $0 < \epsilon < 1$.

2.2. Abelian varieties and the Fourier-Mukai transform. Here we recall some facts about the Fourier-Mukai transform introduced in [Mukai81]. Let \hat{X} be the dual abelian variety and P be the normalized Poincaré line bundle on $X \times \hat{X}$. We denote by $\mathbf{R}\hat{S} : \mathbf{D}(X) \rightarrow \mathbf{D}(\hat{X})$ the usual Fourier-Mukai functor given by $\mathbf{R}\hat{S}(\mathcal{F}) = \mathbf{R}p_{\hat{X}*}(p_X^* \mathcal{F} \otimes P)$. There is a corresponding functor $\mathbf{R}S : \mathbf{D}(\hat{X}) \rightarrow \mathbf{D}(X)$ such that

$$\mathbf{R}S \circ \mathbf{R}\hat{S} = (-1_X)^*[-g] \quad \text{and} \quad \mathbf{R}\hat{S} \circ \mathbf{R}S = (-1_{\hat{X}})^*[-g].$$

Let A be any ample line bundle on \hat{X} , then $\mathbf{R}^0 S(A) = \mathbf{R}S(A)$ is a vector bundle on X of rank $h^0(A)$ which we denote by \hat{A} . For any $x \in X$, let $t_x : X \rightarrow X$ be the translation by x and let $\phi_A : \hat{X} \rightarrow X$ be the isogeny determined by $\phi_A(\hat{x}) = t_{\hat{x}}^* A - A$, then $\phi_A^*(\hat{A}) = \bigoplus_{h^0(A)} A^\vee$.

If (X, Θ) is a PPAV, then $\phi_\Theta : (X, \Theta) \rightarrow (\hat{X}, \hat{\Theta} = \phi_\Theta(\Theta))$ is an isomorphism of PPAVs. If $A = \mathcal{O}_X(m\hat{\Theta})$ for some positive integer $m > 0$, then $\phi_A : \hat{X} \rightarrow X$ can be identified with $m_X : X \rightarrow X$ (multiplication by m) and so it has degree $m^{2 \dim X}$ and $\phi_A^* \Theta \equiv m^2 \hat{\Theta}$. (Note that the above notation is customary, but somewhat confusing as $\mathcal{O}_{\hat{X}}(-\hat{\Theta}) = \mathbf{R}^0 \hat{S}(\mathcal{O}_X(\Theta))$.)

We will need the following result.

Proposition 2.1. *Let \mathcal{F} be a non-zero coherent sheaf on X such that $H^i(\mathcal{F} \otimes P_{\hat{x}}) = 0$ for all $i > 0$ and all $\hat{x} \in \hat{X}$ (where for any $\hat{x} \in \hat{X}$, we let $P_{\hat{x}} = P|_{X \times \hat{x}}$). If $\mathcal{F} \rightarrow k(x)$ is a surjective morphism for some*

$x \in X$, then the induced map $H^0(\mathcal{F} \otimes P_{\hat{x}}) \rightarrow H^0(k(x) \otimes P_{\hat{x}}) \cong k(x)$ is surjective for general $\hat{x} \in \hat{X}$.

Proof. (Cf. [CH03, 2.3] or [PP08].) By cohomology and base change, one sees that $\hat{\mathcal{F}} = \mathbf{R}^0 S(\mathcal{F}) = \mathbf{R}S(\mathcal{F})$ is a sheaf and since $\mathbf{R}\hat{S}(\hat{\mathcal{F}}) = (-1_X)^* \mathcal{F}[-g] \neq 0$, we have that $\hat{\mathcal{F}} \neq 0$. Let $P_x = P|_{x \times \hat{X}}$, then $P_x = \mathbf{R}^0 S(k(x)) = \mathbf{R}S(k(x))$ and the homomorphism $\phi : \hat{\mathcal{F}} \rightarrow P_x$ is non-zero. However, as P_x is a line bundle (and hence torsion free of rank 1), it follows that ϕ is generically surjective. The proposition now follows since for any $\hat{x} \in \hat{X}$, the corresponding fiber of $\hat{\mathcal{F}}$ (resp. $\mathbf{R}S^0(k(x))$) is $H^0(\mathcal{F} \otimes P_{\hat{x}})$ (resp. $H^0(k(x) \otimes P_{\hat{x}}) = H^0(k(x))$). \square

3. MAIN RESULT

Proof of (1.1). Let $0 < \epsilon < 1$ be a rational number such that the index of $\Delta = \frac{1-\epsilon}{m}D$ is not divisible by p . We will show that (X, Δ) is sharply F -pure. Note that for any given m , we may find a sequence of ϵ_i such that the index of $\Delta = \frac{1-\epsilon_i}{m}D$ is not divisible by p and $0 = \lim_{i \rightarrow \infty} \epsilon_i$. Since X is regular, it follows by what we have observed in Subsection 2.1, that $(X, (1-\epsilon)\Delta)$ is strongly F -regular for all rational numbers $0 < \epsilon \leq 1$.

We now fix $e > 0$ such that $(p^e - 1)\Delta$ is Cartier. For any $n > 0$ we have that

$$\phi_{\Delta}^n (F_*^{ne}(\sigma(X, \Delta) \cdot \mathcal{O}_X((1 - p^{ne})(K_X + \Delta)))) = \sigma(X, \Delta).$$

We will show the following.

Claim 3.1. For any sufficiently big integer $n \gg 0$, we have

$$H^i(X, F_*^{ne}(\sigma(X, \Delta) \cdot \mathcal{O}_X((1 - p^{ne})(K_X + \Delta))) \otimes \mathcal{O}_X(\Theta) \otimes P_{\hat{x}}) = 0$$

for all $i > 0$ and all $\hat{x} \in \hat{X}$.

Granting the claim for the time being, we will now conclude the proof of the theorem. Let $x \in X$ be a general point, so that in particular x is not contained in the co-support of $\sigma(X, \Delta)$. We have a surjection

$$F_*^{ne}(\sigma(X, \Delta) \cdot \mathcal{O}_X((1 - p^{ne})(K_X + \Delta))) \otimes \mathcal{O}_X(\Theta) \rightarrow k(x),$$

which factors through $\sigma(X, \Delta) \otimes \mathcal{O}_X(\Theta) \rightarrow k(x)$. By (2.1) and (3.1), we have that

$$H^0(F_*^{ne}(\sigma(X, \Delta) \cdot \mathcal{O}_X((1 - p^{ne})(K_X + \Delta))) \otimes \mathcal{O}_X(\Theta) \otimes P_{\hat{x}}) \rightarrow k(x)$$

is surjective for general $\hat{x} \in \hat{X}$. Since this map factors through $H^0(\sigma(X, \Delta) \otimes \mathcal{O}_X(\Theta) \otimes P_{\hat{x}})$, we have that the induced homomorphism $H^0(\sigma(X, \Delta) \otimes \mathcal{O}_X(\Theta) \otimes P_{\hat{x}}) \rightarrow k(x)$ is surjective. In particular $H^0(\sigma(X, \Delta) \otimes \mathcal{O}_X(\Theta) \otimes P_{\hat{x}}) \rightarrow k(x)$ is surjective.

$P_{\hat{x}} \neq 0$, i.e. the corresponding translate of Θ vanishes along the co-support of $\sigma(X, \Delta)$. But then this co-support is empty so that $\sigma(X, \Delta) = \mathcal{O}_X$. \square

Proof of Claim 3.1. It suffices to show that for all $i > 0$ we have

$$H^i(X, \sigma(X, \Delta) \cdot \mathcal{O}_X((1 - p^{ne})(K_X + \Delta)) \otimes F^{ne*}(\mathcal{O}_X(\Theta) \otimes P_{\hat{x}})) = 0.$$

By Serre vanishing (applied to the projective morphism $p_{\hat{X}} : X \times \hat{X} \rightarrow \hat{X}$ and the coherent sheaf $p_{\hat{X}}^*(\sigma(X, \Delta) \cdot (F^{ne} \times \text{id}_{\hat{X}})^*P)$, we may fix $t > 0$ such that

$$H^i(X, \sigma(X, \Delta) \cdot F^{ne*}P_{\hat{x}} \otimes \mathcal{O}_X(t\Theta)) = 0$$

for $i > 0$. By [PP03, 2.9], it suffices to show that

$$H^i(X, \mathcal{O}_X((1 - p^{ne})(K_X + \Delta) + (p^{ne} - t)\Theta)) = 0$$

for $i > 0$. By assumption $(1 - p^{ne})(K_X + \Delta) + (p^{ne} - t)\Theta \sim_{\mathbb{Q}} ((p^{ne} - 1)\epsilon + 1 - t)\Theta$. The claim now follows since $(p^{ne} - 1)\epsilon + 1 - t > 0$ for $n \gg 0$. \square

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